Lecture 2

Poker
Poker

- **Full house**

- **Straight**

Who wins?
Poker

- Full house

- Straight

Why does full house win?
Poker

- $P(\text{Full house}) = ?$
- $P(\text{Straight}) = ?$
Our plan

1. Language
2. Equally likely outcomes
3. Principle of counting
4. Principle of over-counting
1. Language

- **Outcome:** Head (H), or Tail (T)
- **Sample space (Ω):** set of all possible outcomes: H, T
- It’s fair if \( P(H) = P(T) = \frac{1}{2} \)

- **Event:** a subset of Ω
  - \( P(\{H\}) = P(H) = \frac{1}{2} \) (H is both an outcome and an event)
  - \( P(\{H, T\}) = P(Ω) = 1 \) (\{H,T\} is an event, NOT an outcome)

\( P(A) = P(\text{outcome is in set } A) \)
- \( P(∅) = 0 \), (empty set is an event, NOT an outcome)
Poker

- What is \( \Omega \) for the poker problem?
- A deck of 52 cards
- Picking 5 at random
Poker

- What is \( \Omega \) for the poker problem?
  - A deck of 52 cards
  - Picking 5 at random

- \( S = \{1C, 2C, \ldots, 13C, 1D, 2D, \ldots, 13D, 1H, 2H, \ldots, 13H, 1S, 2S, \ldots, 13S\} \)

- \( \Omega = \{A : A \subseteq S, |A| = 5\} \) (set of sets)
2. Equally likely outcomes

- \( \Omega = \{A : A \subseteq S, |A| = 5\} \)
  - Draw 5 distinct cards uniformly randomly
  - Each outcome in \( \Omega \) is equally likely
2. Equally likely outcomes

- $P(\text{Full house}) = \frac{\text{# of outcomes that are Full house}}{\text{total # of outcomes in } \Omega}$

- $P(\text{Straight}) = \frac{\text{# of outcomes that are Straight}}{\text{total # of outcomes in } \Omega}$
3. Principle of Counting

- total # of outcomes in $\Omega$
- $\Omega = \{A : A \subseteq S, |A| = 5\}$

First, an easier problem:

If we order the 5 cards selected, how many orderings are there?

- How many possibilities for the 1\textsuperscript{st} card?
- How many possibilities for the 2\textsuperscript{nd} card? ......
- $52 \times 51 \times 50 \times 49 \times 48$
3. Principle of Counting

- **Principle of counting:**
  
  \( m \) ways to select one variable, \( n \) ways to select another variable, two selections are independent, then a total of \( mn \) ways to make the pair of selections.

- First, an easier problem:
  
  If we order the 5 cards selected, how many orderings are there?

- How many possibilities for the 1\text{st} card?
- How many possibilities for the 2\text{nd} card? ......
- \( 52 \times 51 \times 50 \times 49 \times 48 \)
4. Principle of Over-counting

- total # of outcomes in $\Omega$
- $\Omega = \{A : A \subseteq S, |A| = 5\}$

- No ordering?

- Principle of Over-counting

With ordering, we are counting A, 3, 4, K, 5 and K, 3, 4, A, 5 as two different hands. Obviously, we are over-counting, so we divide $52 \times 51 \times 50 \times 49 \times 48$ by the number of times we over-count each hand.
4. Principle of Over-counting

- The difference between *ordering*

and *no ordering*
4. Principle of Over-counting

- *ordering* over-counted by 2
4. Principle of Over-counting

- The difference between ordering
4. Principle of Over-counting

and *no ordering*

Over-counted by 3x2
4. Principle of Over-counting

- *total # of outcomes in* \( \Omega \)
- \( \Omega = \{ A : A \subseteq S, |A| = 5 \} \)

- No ordering
- 52 x 51 x 50 x 49 x 48 is over-counting
- By how much?
4. Principle of Over-counting

- For a given set of 5 cards, how many ordering are there?

- This is how much we over-counted.

- Ans: \[
\frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2}
\]
Language

- \( n! = n(n-1)(n-2)\ldots 1 \) \hspace{1cm} (n factorial)

\[
\frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = \frac{52!}{47! \cdot 5!} = \binom{52}{5}
\]

52 choose 5

- \[
\frac{n!}{k!(n-k)!} = \binom{n}{k}
\]

- When no ordering is considered, use \( \binom{n}{k} \)
Poker

- $P(\text{Full house}) = \frac{\# \text{ of outcomes that are Full house}}{\binom{52}{5}}$

- $P(\text{Straight}) = \frac{\# \text{ of outcomes that are Straight}}{\binom{52}{5}}$
Counting

- # of outcomes that are Straight

- Straight only concerns numbers
- It has 5 distinct, consecutive numbers
Counting

- # of outcomes that are Straight

- If we consider only numbers on the 5 cards, how many consecutive five-number sequences?

- 10
Counting

- # of outcomes that are Straight

- Now let’s add the suit

- For each number, how many choices of suit?

- 4
Counting

- # of outcomes that are Straight

- Now let’s add the suit

- We have 5 distinct numbers, how many choices of suit in total?

- $4 \times 4 \times 4 \times 4 \times 4 = 4^5$
Counting

- # of outcomes that are Straight

- 10 ways to pick numbers, $4^5$ ways to pick suits

- # of outcomes that are Straight $= 10 \times 4^5$

- Principle of Counting
Poker

- $P(\text{Full house}) = \frac{\text{# of outcomes that are Full house}}{\binom{52}{5}}$

- $P(\text{Straight}) = \frac{10 \times 4^5}{\binom{52}{5}} = 0.0039$
Last remaining problem

- # of outcomes that are Full house

- First pick numbers, then pick suits
Last remaining problem

- # of outcomes that are Full house

- How many distinct numbers are there in Full house?
- 2 distinct numbers
Last remaining problem

- # of outcomes that are Full house

- Picking the number: *ordering* or *no* ordering?

  - ordering
Last remaining problem

- 1\textsuperscript{st} number for 3 cards, 2\textsuperscript{nd} number for 2 cards
- J followed by A
- A followed by J
Last remaining problem

- Does it make a difference if I give 1\textsuperscript{st} number to 2 cards and 2\textsuperscript{nd} number to 3 cards?

- J followed by A

- A followed by J
Last remaining problem

- How many ways to pick the first number?
  - 13

- How many ways to pick the second number?
  - 12

- How many orderings of 2 distinct numbers?
  - $13 \times 12$
Last remaining problem

- # of outcomes that are Full house
- Pick suits for the 3 cards (with the same number)

- $4^3$ ?
- There are only 4 cards of the same number
- Does ordering matter?
- No. Principle of over-counting. $\binom{4}{3}$
Last remaining problem

- # of outcomes that are Full house
- Pick suits for the 2 cards
  - $\binom{4}{2}$
Last remaining problem

- # of outcomes that are Full house

- Combining numbers and suits (principle of counting)

- $13 \times 12 \times \binom{4}{3} \binom{4}{2}$
Poker

- \( P(\text{Full house}) = \frac{13 \times 12 \times \binom{4}{3} \binom{4}{2}}{\binom{52}{5}} = 0.0014 \)

- \( P(\text{Straight}) = \frac{10 \times 4^5}{\binom{52}{5}} = 0.0039 \)