Lecture 5

Polling (advanced)

What can we infer from an observation?

Review discrete r.v.
### Bernoulli process

Coin toss 1, coin toss 2, coin toss 3, coin toss 4 ...

\[ Z_1 \quad Z_2 \quad Z_3 \quad Z_4 \]

\[ Z_i = \begin{cases} 0, & \text{with probability } 1 - p \\ 1, & \text{with probability } p \end{cases} \]

\[ P(Z_1 = 1, Z_2 = 0, Z_3 = 1, Z_4 = 1) = ? \]

\[ p(1-p)p = p^3(1-p) \]

i.i.d. : independent and identically distributed

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### Problem solving 1

- Roll a fair die 10 times. Let \( X \) be the number of times the die shows one or two. What’s the mean of \( X \)?

- What is the Bernoulli process here?

<table>
<thead>
<tr>
<th>Number that shows on the die</th>
<th>{1, 2}</th>
<th>{3, 4, 5, 6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_i )</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ Z_i = \begin{cases} 0, & \text{w.p } 2/3 \\ 1, & \text{w.p } 1/3 \end{cases} \]

What is \( X \) in terms of \( Z_i \)?
Problem solving 1

- Roll a fair die 10 times. Let $X$ be the number of times the die shows one or two. What’s the mean of $X$?

$$Z_i = \begin{cases} 0, & w.p \ 2/3 \\ 1, & w.p \ 1/3 \end{cases}$$

Keyword: number of times

$$X = \sum_{i=1}^{10} Z_i$$

What’s the distribution of $X$?

\[ X \sim \text{Binomial} \ (10, \ 1/3) \]
### Problem solving 2

- Roll a fair die. Let $Y$ be the number of trials needed until the die shows one or two. What’s the mean of $Y$?

- What is the Bernoulli process here?

<table>
<thead>
<tr>
<th>Number that shows on the die</th>
<th>{1, 2}</th>
<th>{3, 4, 5, 6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_i$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$$Z_i = \begin{cases} 
0, & \text{w.p. } 2/3 \\
1, & \text{w.p. } 1/3 
\end{cases}$$

What’s the distribution of $Y$?

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- Roll a fair die 10 times. Let $Y$ be the number of trials needed until the die shows one or two. What’s the mean of $Y$?

$$Z_i = \begin{cases} 
0, & \text{w.p. } 2/3 \\
1, & \text{w.p. } 1/3 
\end{cases}$$

**Keyword:** number of trials until

$Y \sim \text{Geometric (1/3)}$
Problem solving 3

- Roll a fair die. Let $S$ be the number of trials needed until the die shows one or two for the second time. What’s the mean of $S$?

$$Z_i = \begin{cases} 0, & \text{w.p. } 2/3 \\ 1, & \text{w.p. } 1/3 \end{cases}$$

**Keyword:** number of trials until second time

If $L_1$ is the number of trial till the first 1, $L_2$ is the number of trials till the second 1, then $S = L_1 + L_2$.

Here $L_1$ and $L_2$ are i.i.d. Both $\sim$ Geometric $(1/3)$
Problem solving 4

- Coupon collector

Suppose a fair die is repeatedly rolled until each of the numbers one through six shows at least once. What is the mean number of rolls?

**Keyword: number of trials until**

By same argument,
if \( L_1 \) is the number of trials till the first 1, \( L_2 \) is the number of trials till the second 1, ... then \( S = L_1 + L_2 + L_3 + L_4 + L_5 + L_6 \)

However, the underlying Bernoulli process changes!!
Problem solving 4

- L₁ number of trials until any number shows up.

<table>
<thead>
<tr>
<th>Number that shows on the die</th>
<th>{1, 2, 3, 4, 5, 6}</th>
<th>φ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z₁</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ Z_i = \begin{cases} 
0, & \text{w. p. 0} \\
1, & \text{w. p. 1} 
\end{cases} \]

- In particular, Z₁ can only be 1.
- So L₁ = 1

Problem solving 4

- Bernoulli process so far

- L₂ starts counting
- Suppose the number that showed up is 6

<table>
<thead>
<tr>
<th>Number that shows on the die</th>
<th>{1, 2, 3, 4, 5}</th>
<th>{6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zᵢ (i &gt; L₁)</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ Z_i = \begin{cases} 
0, & \text{w. p. 1/6} \\
1, & \text{w. p. 5/6} 
\end{cases} \]

What’s the mean of L₂?
Problem solving 4

- Bernoulli process so far
  1 0 0 1
- \( L_3 \) starts counting
- Suppose the two numbers that showed up is 6 and 5

<table>
<thead>
<tr>
<th>Number that shows on the die</th>
<th>{1, 2, 3, 4}</th>
<th>{5, 6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_i(i &gt; L_2) )</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
Z_i = \begin{cases} 
0, & \text{w. p. } 2/6 \\
1, & \text{w. p. } 4/6 
\end{cases}
\]

What’s the mean of \( L_3 \)?

Problem solving 4

\[
S = L_1 + L_2 + L_3 + L_4 + L_5 + L_6
\]

- \( L_1 \sim \text{Geometric (1)} \)
- \( L_2 \sim \text{Geometric (5/6)} \)
- \( L_3 \sim \text{Geometric (4/6)} \)
- \( L_4 \sim \text{Geometric (3/6)} \)
- \( L_5 \sim \text{Geometric (2/6)} \)
- \( L_6 \sim \text{Geometric (1/6)} \)

\[
E(S) = E(L_1) + E(L_2) + E(L_3) + E(L_4) + E(L_5) + E(L_6)
\]

\[
= 1 + 6/5 + 6/4 + 6/3 + 6/2 + 6
\]

\[
= 6 \left( 1/6 + 1/5 + \frac{3}{6} + 1/3 + \frac{3}{6} + 1 \right)
\]

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What can we infer from an observation?

180 m votes for C
120 m votes for D
Polling

- Pick one person: (random experiment)
  - With 0.4 probability, vote for D
  - With 0.6 probability, vote for C
  - Bernoulli (p)

- Question now:
  If we don’t know p, what can we say about its value based on the result of our random experiment?
  (In this case, outcome of a Bernoulli process)
Inference

This week
What can we say about p?
Values of random variables observed

3 different ways of making sense of the observation

1. Maximum likelihood (ML) parameter estimation
2. Bayes formula
3. Confidence interval

1. ML parameter estimation

Given a bent coin that shows heads with probability p. I flip the coin 1000 times and get 300 flips. How do I estimate p?

p = 0.3, the empirical mean

0.3 is the ML estimate in the following sense:

\[ P_x(300) = \binom{1000}{300} p^{300} (1 - p)^{700} \]
1. **ML parameter estimation**

0.3 is the ML estimate in the following sense

\[ p_{ML} = \operatorname{argmax}_p \binom{n}{k} p^k (1 - p)^{n-k} \]

Differentiate with respect to \( p \)

1. Identify the variable that parametrizes the distribution (in this case, \( p \))

2. Maximize the distribution with respect to the parameter
2. Bayes formula

We have *prior knowledge* of the distribution of $p$

$p = 0.3$, the empirical mean

However, we *know* that $p = 0.1$ with probability $0.9$ and $p = 0.3$ with probability $0.1$
2. Bayes formula

\[ P(p = 0.1) = 0.9, \quad P(p = 0.3) = 0.1 \]

Suppose
\[ P(k = 300 \mid p = 0.1) = 0.1 \]
\[ P(k = 300 \mid p = 0.3) = 0.5 \]

What is \( P(p = 0.1 \mid k = 300) \)?

How about \( P(p = 0.3 \mid k = 300) \)?

Bayes formula is nothing but conditional probability

\[ P(p = 0.1 \mid k = 300) = \frac{P(p = 0.1, k = 300)}{P(k = 300)} \]

\[ P(p = 0.1) = 0.9 \quad P(p = 0.3) = 0.1 \]
2. Bayes formula

\[
P(p = 0.1 | k = 300) = \frac{P(p = 0.1, k = 300)}{P(k = 300)}
\]

\[
= \frac{P(p = 0.1, k = 300)}{P(p = 0.1, k = 300) + P(p = 0.3, k = 300)}
\]

\[
P(p = 0.1) = 0.9, \quad P(p = 0.3) = 0.1
\]

Suppose \( P(k = 300 | p = 0.1) = 0.1 \)

\( P(k = 300 | p = 0.3) = 0.5 \)
2. Bayes formula

\[
P (p = 0.1 \mid k = 300) = \frac{P (p = 0.1, k = 300)}{P (k = 300)} = \frac{P (p = 0.1, k = 300)}{P (p = 0.1, k = 300) + P (p = 0.3, k = 300)}
= \frac{0.9 \times 0.1}{0.9 \times 0.1 + 0.1 \times 0.5} = \frac{9}{14}
\]

Suppose \( P (k = 300 \mid p = 0.1) = 0.1 \)
\( P (k = 300 \mid p = 0.3) = 0.5 \)

\[
P (p = 0.1) = 0.9, \quad P (p = 0.3) = 0.1
\]

Similarly, we get
\[
P (p = 0.3 \mid k = 300) = \frac{5}{14}
\]
3. Confidence interval

ML parameter maximization or Bayes formula
Yields a point estimate: \( p = 0.1? \ p = 0.3? \)

Confidence interval gives an interval estimate, so that
\[ P \left( p \in \text{this interval} \right) \geq \text{some given number} \]

Each set of samples gives a different random interval

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There is a distribution on the random interval. The probability that a random interval contains \( p \) is made to be at least, say, 96%